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How is turbulent energy dissipated in a superfluid?

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Abstract
At a sufficiently large Reynolds number the flow of a classical fluid becomes turbulent. Typically, turbulent energy is injected into large-scale eddies, from which it flows through the action of the non-linear term in the Navier–Stokes equation into smaller and smaller eddies until the scale of the motion is small enough that the energy can be dissipated by viscosity. In a superfluid (liquid ^{4}He or liquid ^{3}He), which has no viscosity, this source of dissipation is absent, so that perhaps the flow of turbulent energy persists down to atomic scales; furthermore, the absence of viscosity seems to imply that the Reynolds number is infinite. However, turbulent motion in a superfluid is restricted by quantum effects, associated with the quantization of angular momentum, and it will be shown how these effects change this picture and lead to an effectively finite Reynolds number, with dissipation that mimics the effect of viscosity and arises from the radiation of sound by special types of quantized motion.

(Some figures in this article are in colour only in the electronic version)

1. Introduction: classical turbulence

An understanding of turbulence in a superfluid depends on some understanding of turbulence in a classical fluid [1]. This latter case is based on the Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}, \tag{1}$$

where p is the pressure and ν the kinematic viscosity. We note the presence of the non-linear inertial term, $(\mathbf{v}\cdot\nabla)\mathbf{v}$, and the term, $\nu\nabla^2\mathbf{v}$, describing the dissipative effect of viscosity. Turbulent solutions of the Navier–Stokes equation exist only if the Reynolds number, Re, is sufficiently large. This dimensionless number is a measure of the ratio of the magnitude of the non-linear term to that of the dissipative term; for a characteristic velocity U and a characteristic length scale R, it can be written as UR/ν. In a typical turbulent flow the shear generated by flow past a boundary leads to the injection of energy in the form of large eddies, with a size determined by the characteristic dimensions of the boundary. As long as the Reynolds

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number associated with these eddies is large compared with unity, viscosity has a negligible effect. However, the eddies do not persist, because the non-linear term in the Navier–Stokes equation leads to a coupling between motion on different length scales and energy is therefore flows from the large eddies to smaller eddies. It is believed that this transfer takes place in a cascade (the Richardson cascade), from large eddies to slightly smaller eddies, and then to slightly smaller eddies again, and so on. This process continues, with conservation of energy, until the Reynolds number of the smallest eddies falls to unity, when the energy is lost by viscous dissipation, and the cascade is terminated. The character of the turbulence is strongly influenced by the range of eddy sizes involved and therefore by the Reynolds number of the largest eddies, which is determined by the characteristic length associated with the obstacle inducing the turbulence and the characteristic flow velocity past this obstacle. It is interesting to ask how the turbulent energy is distributed over different length scales in the Richardson cascade, or equivalently over different wavenumbers in a spatial Fourier analysis of the velocity field. In suitably simple cases, such as flow through a grid, the turbulence is approximately homogeneous and isotropic, and this distribution can be written in terms of an energy spectrum $E(k)$, such that the energy per unit mass associated with wavenumbers having magnitude in a range $\mathrm{d}k$ is $E(k)\,\mathrm{d}k$. Over the range of wavenumbers where energy is conserved (the *inertial range*) this spectrum has, approximately, the *Kolmogorov form*

$$E(k) = C\epsilon^{2/3}k^{-5/3}, \tag{2}$$

where C is a constant of order unity and ϵ is the constant rate at which energy per unit mass flows down the cascade. The cascade is terminated by viscosity at a wavenumber of order the *Kolmogorov dissipation wavenumber*, given by

$$k_{\mathrm{d}} = \epsilon^{1/4}\nu^{-3/4}. \tag{3}$$

For semiquantitative discussions it is often useful to derive from equation (2) a quantity that we can loosely describe as the mean square velocity associated with eddies of size r, and which can be written

$$v_r^2 \sim \epsilon^{2/3}r^{2/3}. \tag{4}$$

The rate of viscous dissipation of energy in homogeneous turbulence, taking place around the wavenumber (3), can be written

$$\epsilon = \nu\langle\omega^2\rangle, \tag{5}$$

where $\langle\omega^2\rangle$ is the mean square vorticity in the fluid.

2. Superfluids

Superfluidity [2] is associated with the formation in a fluid of a *Bose condensate*, in which a macroscopic fraction of the fluid particles, assumed to be Bosons, exists in a single quantum state. It was discovered first in the liquid phase of the heavier isotope of helium, ^{4}He, where it exists at temperatures less than about 2.2 K. Much more recently it has been found in Bose-condensed gases, produced by laser and evaporative cooling techniques at temperatures below about 100 nK. In a more subtle way it can exist also in a fluid of Fermi particles, such as liquid ^{3}He, in which the particles undergo *Cooper pairing*, the Cooper pairs then undergoing a form of Bose condensation; this type of Bose condensation of Cooper pairs of electrons is responsible for superconductivity.

At a finite temperature superfluids exhibit *two-fluid* behaviour, in which a *normal fluid*, behaving like a conventional viscous fluid, coexists with a *superfluid component*, which is able to flow without viscous dissipation. The normal fluid is composed of thermal excitations; the superfluid component is formed from what remains of the ground state, including the Bose condensate. The lack of viscous dissipation in the superfluid component is connected with the Bose condensation, superfluid flow being associated with flow of this condensate.

In this paper we shall be concerned with the behaviour of a superfluid at a very low temperature, when the density of normal fluid is so small that it can be neglected, so that the whole fluid can flow without viscous dissipation. More specifically, we shall describe the behaviour of such a superfluid when it is turbulent. It turns out that flow of the superfluid can indeed be turbulent. Furthermore, on large enough length scales there is likely to be an essentially classical Richardson cascade in which energy flows from large-scale eddies to successively smaller eddies [3]. Given that there can be no viscous dissipation, we are led to ask whether dissipation does in fact occur, and if so, how, and on what length scale. Perhaps dissipation can set in only on atomic scales, resulting in fluid flows with Reynolds numbers that are very much larger than is possible in a conventional viscous fluid. As we shall show, however, the possibility of flow with such extreme Reynolds numbers is not realized, owing to the intervention on small length scales of quantum effects, to which we now turn our attention.

2.1. Quantum restrictions on superfluid flow

As we have suggested, superfluid flow is closely linked to flow of the condensate. The condensate is characterized by a macroscopic number of particles in a single quantum state. Such a situation is found also for photons in a laser, where it gives rise to the formation of a coherent electromagnetic field. Superfluidity is associated with the formation of a similar coherent field, but now a coherent *particle field*, formed, in the case of ^{4}He, from the helium atoms, and superfluid flow at velocity v_s arises when there is a gradient in the phase (S) of the wavefunction into which condensation has taken place, as in ordinary quantum mechanics; more precisely

$$v_s = \frac{\hbar}{m_4}\nabla S, \tag{6}$$

where m_4 is the mass of a helium atom. The coherent particle field, or *condensate wavefunction*, must be single valued, and this leads to a quantum condition on the hydrodynamic circulation for any closed circuit in the helium

$$\oint \mathbf{v_s}\cdot \mathbf{d}\ell = n\kappa, \tag{7}$$

where n is an integer and $\kappa = 2\pi\hbar/m_4$ is the *quantum of circulation*; clearly this condition amounts essentially to the quantization of angular momentum.

It follows from equations (6) and (7) that rotational motion in a simply connected volume of the superfluid can exist only in the form of *quantized vortex lines*, round which there is an irrotational circulation given by equation (7), in practice with $n = 1$ (figure 1). Along the line itself superfluidity must be destroyed, to avoid an infinite superfluid velocity, the destruction extending over a core, which for ^{4}He is of atomic size; away from the core the velocity falls off with distance as $1/r$. Vortex lines must either end on a solid boundary or form closed curves in the liquid. If placed in a vessel rotating at a steady angular velocity Ω, the superfluid mimics uniform rotation as best it can by becoming filled with a uniform array of vortex lines directed along Ω with density $2\Omega/\kappa$. As is easily seen, the flow on a scale large compared with the vortex spacing is in fact then closely similar to uniform rotation. An analogous situation is found in a turbulent superfluid.

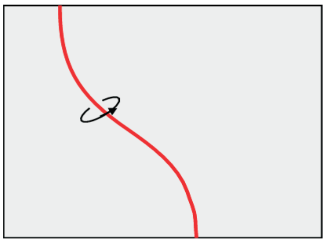

Figure 1. A quantized vortex line.

3. Quantum turbulence at very low temperatures

As we have mentioned, flow of the superfluid component can become turbulent, a form of turbulence now often called *quantum turbulence* [3]. The rotational motion characteristic of turbulent flow must take the form of some more or less irregular configuration of quantized vortex lines, often called a *vortex tangle*.

We note that according to equation (4) the average circulation associated with classical eddies of size r in the inertial range must be given by

$$\Gamma \sim 2\pi\epsilon^{1/3}r^{4/3}. \tag{8}$$

If $\Gamma \gg \kappa$, we can think of the eddy of size r as of necessity containing many quanta of circulation. It seems reasonable to suppose that, if this condition holds, the eddies behave classically, as we have suggested, but if it does not we can expect serious departures from classical behaviour. The condition

$$\ell \sim \kappa^{3/4}\epsilon^{-1/4} \tag{9}$$

defines the length scale at which there is the change of behaviour: for $r \gg \ell$ we expect classical behaviour; for $r \ll \ell$ we expect quantum behaviour.

To proceed further we must take account of the fact that quantum turbulence involves configurations of quantized vortex lines. It follows from equations (4) and (9) that $\ell \sim \kappa v_\ell^{-1}$. We assume for the moment that the vortex lines are more or less evenly spaced and have curvatures of order the reciprocal of this spacing (a 'smooth tangle': figure 2); in other words there is no structure on scales significantly less than the average spacing, except for the vortex cores. In this case ℓ is seen to be simply the average spacing between the lines, as we see from the fact that the velocity at a distance ℓ from an isolated vortex is of order κ/ℓ. Our conclusion that classical behaviour can be expected on length scales much larger than this spacing, but not on smaller scales, has been verified by direct computer simulations [4], which have demonstrated the existence of an inertial-range Kolmogorov spectrum, provided, of course, that there is some source of dissipation at small length scales. There can be no Kolmogorov spectrum without this dissipation.

Experiments at very low temperatures that are directly relevant have not yet been completed. Experiments at higher temperatures do indicate the existence of a Kolmogorov spectrum [3], but dissipation is then provided by the normal fluid, as described in the next section. The rest of our discussion is therefore theoretical speculation, which remains to be verified experimentally.

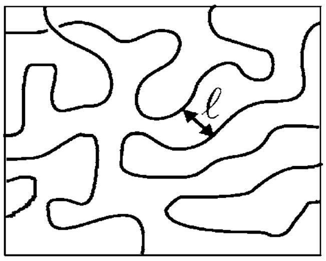

Figure 2. Smooth vortex tangle with vortex spacing ℓ.

4. Dissipation in quantum turbulence at very low temperatures

We come then to the question posed in section 2: what mechanism can provide dissipation at very low temperatures, and what is the effective Reynolds number for the quantum turbulence?

We remark first that, at temperatures where there is a significant fraction of normal fluid, two dissipative processes are known to operate: first there can be viscous dissipation in the normal fluid; and secondly the motion of a vortex relative to the normal fluid results in a dissipative drag on the vortex [3]. Both these processes cease to operate at the lowest temperatures. At first sight there is no mechanism for dissipation at the lowest temperatures, and there seems to be the possibility that a Kolmogorov spectrum may not exist.

That this possibility can be ruled out follows from a simple practical observation: vortices can generate sound, as is clear from the noise produced by an aircraft. It is easily shown that a vortex (quantum or not), oscillating in position, leads to an oscillating pressure, which leads in turn to the radiation of sound (recall that towards the centre of a vortex there is a reduction in pressure from the Bernoulli effect). The intensity of the radiation increases strongly with increasing frequency. In a random homogeneous configuration of quantum vortices with spacing ℓ the frequency with which the vortices oscillate in position is of order κ/ℓ^2. This frequency proves to be much too small for effective radiation of sound, so that we are led to the conclusion that on length scales of order or greater than ℓ there is no effective dissipation [3].

The only way in which the required high frequencies can be produced lies in the production of vortex structures that are much smaller than ℓ, contrary to the assumption we made in the preceding section. The fact that such structures might form is not surprising, since our experience with the Richardson cascade suggests that, in the absence of dissipation, turbulent energy will flow into smaller and smaller structures. We ask what form these small structures take, and in what ways they can radiate sound [3].

Computer simulations [5] show that the evolution of the turbulent vortex tangle leads occasionally to the close approach of two vortices, this alone producing a local small-scale structure. More importantly, such simulations, especially those using a model superfluid described by the non-linear Schrödinger equation [6], show that this close approach is frequently accompanied by a *vortex reconnection*, as shown in figure 3. Immediately after such a reconnection both vortices have on them a sharp kink, which serves as the source of waves on the vortices, in much the same way as the plucking of a stretched string generates waves on the string. The waves on a vortex are *Kelvin waves* [7]: they are helical waves with the approximate dispersion relation

$$\omega = \frac{\kappa k^2}{4\pi} \ln\left(\frac{1}{k\xi}\right), \qquad (10)$$

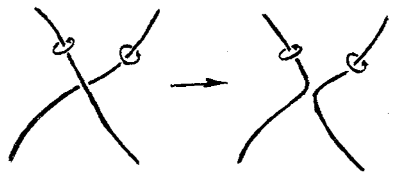

Figure 3. A vortex reconnection.

where ξ is the radius of the vortex core. The Kelvin waves can radiate sound [8]. However, the range of Kelvin-wave frequencies produced directly by a single reconnection does not extend to sufficiently high values with sufficient intensity to produce significant dissipation. However, *repeated reconnections* can lead to a build-up of the Kelvin-wave amplitudes, until eventually non-linear interactions lead to the generation of higher frequencies. It seems that this process of transfer of Kelvin-wave energy to higher and higher frequencies takes place in a kind of cascade, analogous to the Richardson cascade. Eventually we can expect the energy to reach frequencies of order 1 GHz, where efficient sound radiation can take place.

The reconnection process itself, which can be described only quantum mechanically, also leads to some radiation of sound. The resulting energy loss in ^{4}He is quite small and can probably be neglected, although it is likely to be much larger in cases where the vortex core is larger, notably in Bose-condensed gases.

Details of the probable characteristics of the Kelvin-wave cascade have come from computer simulations. Once the amplitude of the waves has become sufficiently large, the associated steady-state energy spectrum seems to have the rough form [9]

$$\tilde{E}(\tilde{k}) \sim A\rho\kappa^2\tilde{k}^{-1}, \qquad (11)$$

independent, roughly, of the energy flow $\tilde{\epsilon}$, where $\tilde{E}(\tilde{k})\,\mathrm{d}\tilde{k}$ is the Kelvin-wave energy per unit length of vortex in the range $\mathrm{d}\tilde{k}$ of Kelvin wavenumbers (a spectrum that is probably more accurate has been proposed in [10]).

Thus we can summarize the probable structure of homogeneous quantum turbulence in liquid ^{4}He at a very low temperature as follows (figure 4). The turbulence takes the form of a more or less random configuration of quantized vortex lines. On length scales greater than the average spacing between these lines, the fluid dynamics is relatively unaffected by quantum effects, and an inertial-range Richardson–Kolmogorov cascade is formed, in which energy is carried towards smaller length scales without dissipation. On scales less than the average line spacing, quantum effects become dominant, and energy is carried to smaller length scales by a Kelvin-wave cascade, in which there is no dissipation until the Kelvin-wave frequency is so high that effective energy loss by radiation of sound becomes possible, when the Kelvin-wave cascade is terminated.

We comment in passing that at very low temperatures the thermal excitations in superfluid ^{4}He are phonons, or quantized sound waves. Thus the dissipation that we are describing involves simply the conversion of turbulent energy into heat, as in any viscous process.

We turn finally to the rate at which dissipation occurs, and to the question whether the turbulent superfluid is characterized by some effective Reynolds number. It is convenient now to introduce two lengths: the length, L, of vortex line per unit volume; and the length, L_0, of vortex line after smoothing to remove the Kelvin waves. It can be shown that these two lengths differ by a factor that is only weakly dependent on L and typically only a little larger than unity [3]. The average spacing, ℓ, between the smoothed lines is equal to $L_0^{-1/2}$. From

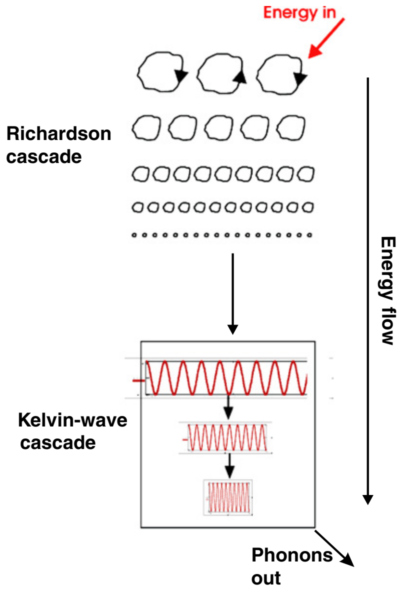

Figure 4. The Richardson and Kelvin-wave cascades.

equation (9) we find then that

$$\epsilon \sim \kappa^3 \ell^{-4} = \kappa^3 L_0^2 = A\kappa(\kappa L)^2, \tag{12}$$

where the dimensionless factor A is a little larger than unity. The quantity $(\kappa L)^2$ has the dimensions of vorticity squared, and it is clear that, since all vorticity in the superfluid is confined to the cores of vortex lines, this quantity must be some measure of the mean square vorticity in the fluid (see [11] for a full discussion). We can therefore compare equation (12) with equation (4), to find that the turbulent superfluid has an effective kinematic viscosity given by

$$\nu' = A\kappa. \tag{13}$$

Thus the turbulent superfluid is behaving as far as dissipation is concerned like a classical fluid with an effective kinematic viscosity equal to roughly the quantum of circulation. As a result of what is probably a numerical accident, this value of the kinematic viscosity is not very different from that of the normal phase of liquid ^{4}He. Thus in a possible quest for a very high Reynolds number in turbulent flow nothing seems to be gained by cooling the helium from the normal phase to a very low temperature in the superfluid phase. This conclusion holds in spite of the fact that the superfluid has zero viscosity: our analysis suggests that the quantization conditions governing turbulent flow in the superfluid phase have the curious effect of introducing dissipation at a rate similar to that in the normal phase. It remains to be verified experimentally whether this is indeed the case.

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